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AXISYMMETRIC BENDING OF A HEATED CIRCULAR PLATE ON AN ELASTIC BASE WITH ACCOUNT OF ITS DEFORMABILITY OVER ITS THICKNESS

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The axisymmetric problem is solved for the bending of a circular plate on a heated half-space under the action of a distributed load and a temperature field.

One of the founders of the theory of bending of beams and plates on an elastic base is Proktor [1, 2], who, in 1919, formulated a computational process for the reduction of the problem of bending of a narrow beam on a half-space to the solution of an integrodifferential equation taking account of the elastic deformations of contiguous bodies. Because the series of solutions that he obtained proved to be weakly convergent [2], another variant of this method was formulated, based on integral account of the crumpling of a beam over its thickness [3, 4]. As was shown by these calculations, taking account of the crumpling of a beam over thickness leads to a considerable redistribution of the reaction pressure under the base of the beam. Below, we give a further development of Proktor's method applied to circular plates resting on an elastic half-space.

1. We consider a circular plate of radius a, on the bounding planes of which the external loads and temperature are constant:

$$\sigma_z = -q,\ T = T_1 = \text{const} \ \text{for}\ z = -h,$$
$$\sigma_z = -p,\ T = T_2 = \text{const} \ \text{for}\ z = h. \qquad (1)$$

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We assume that the temperature varies linearly over the thickness of the plate and is independent of the radial coordinate. To determine the value for the crumpling of a circular plate we must find the normal displacement u_z. On the basis of Hooke's law, taking account of the temperature distribution and the Cauchy equilibrium equations, in a cylindrical coordinate system for the case of axial symmetry, we have the following system of equations for the components of the displacement vector:

$$\mu\left(\Delta u_r - \frac{u_r}{r^2}\right) + (\lambda+\mu)\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] - (3\lambda+2\mu)\alpha\frac{\partial T}{\partial r} = \tag{2}$$

$$\mu\Delta u_z + (\lambda+\mu)\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\right] - (3\lambda+2\mu)\alpha\frac{\partial T}{\partial z} = 0. \tag{3}$$

We seek a particular solution of Eqs. (2) and (3) in the form

$$u_r = \frac{\partial\Psi}{\partial r},\quad u_z = \frac{\partial\Psi}{\partial z}. \tag{4}$$

Substituting (4) into (2) and (3), we obtain

$$-\mu\frac{1}{r^2}\frac{\partial\Psi}{\partial r} + \frac{\partial}{\partial r}[(\lambda+2\mu)\Delta\Psi - (3\lambda+2\mu)\alpha T] = 0, \tag{5}$$

$$\frac{\partial}{\partial z}[(\lambda+2\mu)\Delta\Psi - (3\lambda+2\mu)\alpha T] = 0. \tag{6}$$

A solution of Eqs. (5) and (6) satisfying conditions (1) is the function

$$\Psi = Cz + \frac{3\lambda+2\mu}{2(\lambda+2\mu)}\alpha\left(Az^2 + \frac{1}{3}Bz^3\right), \tag{7}$$

where

$$A = -\frac{1}{2\alpha(3\lambda+2\mu)}(p+q) + \frac{1}{2}(T_2+T_1),$$
$$B = -\frac{1}{2\alpha(3\lambda+2\mu)h}(p-q) + \frac{1}{2h}(T_2-T_1).$$

Convergence of the bounding planes (crumpling) of the plate equals

$$w_{\text{cr}} = u_z(-h) - u_z(h) = \frac{(1-2\nu)(1+\nu)h}{E(1-\nu)}(p+q) - \frac{1+\nu}{1-\nu}\alpha h(T_2+T_1). \tag{8}$$

2. We consider the bending of a freely supported circular heated plate on an elastic base. As usual, we assume that the temperature over the thickness of the plate varies according to a linear law. Then the thermal stresses in the plate equal zero [5]; therefore, the normal deflection of the middle plane of the plate w(r) for an axisymmetric load is determined from the equation

$$\frac{D}{a^4}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 w = q - p. \tag{9}$$

Using the Green's function for the auxiliary boundary-value problem of [6], from (9) we obtain

$$w = \frac{a^4}{D}\int_0^1 G(r,t)[q(t)-p(t)]\,t\,dt + C_0 + C_1r^2, \tag{10}$$

where

$$G(r,t) = \frac{1}{4}\begin{cases}(r^2+t^2)\ln t + 1 - t^2, & r<t,\\ (t^2+r^2)\ln r + 1 - r^2, & t<r.\end{cases}$$

The arbitrary constants C_0 and C_1 are determined from the conditions satisfied by the free edge [7]:

$$\int_0^1 tp(t)\,dt = \int_0^1 tq(t)\,dt,$$

$$\int_0^1 t^3 p(t)\,dt + \frac{16Eh^3}{3a^4(1-\nu)^2} C_1 = \int_0^1 t^3 q(t)\,dt. \tag{11}$$

3. We assume that on the boundary of the half-space $z \geq 0$ in an arbitrary region D we are given a normal load p(x, y) and temperature $T_2(x, y)$; outside the region D, the temperature and load are absent. Then the temperature distribution T(x, y, z) in the half-space is given by

$$T(x, y, z) = \frac{z}{2\pi} \iint_D \frac{T_2(\alpha, \beta)\,d\alpha d\beta}{\sqrt{[(x-\alpha)^2 + (y-\beta)^2 + z^2]^3}}.$$

The stresses and displacements corresponding to the temperature field of the half-space are expressed in terms of the thermoelastic potential of the displacements $\Phi(x, y, z)$, which has the form [8]*

$$\Phi(x, y, z) = \frac{\beta_0}{4\pi} \iint_D \ln \frac{\sqrt{(x-\alpha)^2 + (y-\beta)^2 + z^2} - z}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} T_2(\alpha, \beta)\,d\alpha d\beta - \\ - \frac{\beta_0 z}{4\pi} \iint_D \frac{T_2(\alpha, \beta)\,d\alpha d\beta}{\sqrt{(x-\alpha)^2 + (y-\beta)^2 + z^2}}, \tag{12}$$

where

$$\beta_0 = \frac{1+\nu}{1-\nu}\alpha_0.$$

Thus, the distribution of the normal stresses σ_z in the region D taking the temperature into account is determined by the equation

$$-\sigma_z = p(x, y) - \frac{E\alpha_0}{2(1-\nu)} T_2(x, y).$$

4. The sag (normal displacement) of the boundary points of the half-space in the case under consideration can be represented in the following way:

$$w_0(r) = \frac{4(1-\nu_0^2)}{\pi E_0} \left[\int_0^r \int_0^{\frac{\pi}{2}} \frac{p(t)\,td\varphi dt}{\sqrt{r^2 - t^2 \sin^2\varphi}} + \int_r^1 \int_0^{\frac{\pi}{2}} \frac{p(t)\,td\varphi dt}{\sqrt{t^2 - r^2\sin^2\varphi}} \right] - \\ - \frac{2(1+\nu_0)\alpha_0}{\pi} \left[\int_0^r \int_0^{\frac{\pi}{2}} \frac{T_2(t)\,td\varphi dt}{\sqrt{r^2 - t^2\sin^2\varphi}} + \int_r^1 \int_0^{\frac{\pi}{2}} \frac{T_2(t)\,td\varphi dt}{\sqrt{t^2 - r^2\sin^2\varphi}} \right]. \tag{13}$$

The normal displacement of the points of the middle surface of a plate on an elastic base under the action of an axisymmetric load q(r) can be represented in the form

$$w = w_0 + \frac{1}{2} w_{cr}, \tag{14}$$

where w_0 is the displacement of the lower surface of the plate, equal to the sag of the boundary points of the half-space (13); w_{cr} is the approach of the two bounding surfaces (crumpling) of the plate (8). Equating the right sides of (10) and (14), we obtain the following integral equation of the second kind for the determination of the unknown reaction pressure:

*There is an error in the article [8].

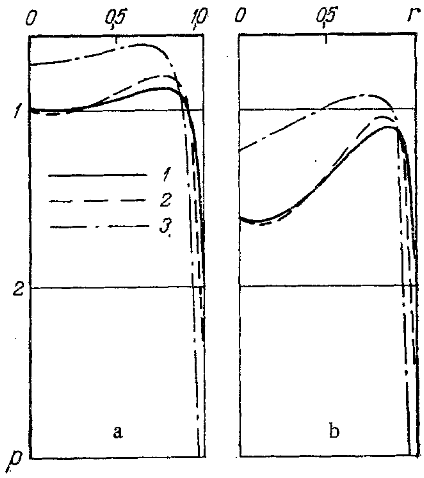

Fig. 1. Distribution of the reaction pressure under a plate loaded by an external load for q(r) = 1 (a) and q(r) = 2 − r (b): 1) reaction pressure with account of crumpling of the plate without temperature; 2) the same, with account of crumpling of a heated plate; 3) the same, neglecting crumpling and temperature.

$$\varepsilon p(r) + \int_0^1 K(r, t)\, p(t)\, t dt = f(r), \tag{15}$$

where

$$K(r, t) = \begin{cases} \beta \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{r^2 - t^2 \sin^2 \varphi}} + \gamma [(r^2 + t^2) \ln r + 1 - r^2], & t < r, \\ \beta \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{t^2 - r^2 \sin^2 \varphi}} + \gamma [(r^2 + t^2) \ln t + 1 - t^2], & r < t, \end{cases} \tag{16}$$

$$f(r) = -\varepsilon q(r) + 4\gamma \int_0^1 G(r, t)\, q(t)\, t dt + \frac{2(1-\nu)E}{(1+\nu)(1-2\nu)a}(C_0 + C_1 r^2) -$$

$$-\eta \left[\int_0^r \int_0^{\frac{\pi}{2}} \frac{T_2(t)\, t d\varphi dt}{\sqrt{r^2 - t^2 \sin^2 \varphi}} + \int_r^1 \int_0^{\frac{\pi}{2}} \frac{T_2(t)\, t d\varphi dt}{\sqrt{t^2 - r^2 \sin^2 \varphi}} \right] + \varepsilon \frac{\alpha E}{1 - 2\nu}(T_1 + T_2),$$

$$\varepsilon = \frac{h}{a}, \quad \beta = \frac{8E(1-\nu)(1-\nu_0)}{\pi E_0 (1-2\nu)(1+\nu)}, \quad \gamma = \frac{3(1-\nu)^2 a^3}{4(1-2\nu)h^3}, \quad \eta = \frac{4E(1-\nu)(1+\nu_0)\alpha_0}{\pi(1-2\nu)(1+\nu)}. \tag{17}$$

Since $T_2(r)$ = const, for the function f(r) we have

$$f(r) = -\varepsilon q(r) + 4\gamma \int_0^1 G(r, t)\, q(t)\, t dt + \frac{2(1-\nu)E}{(1+\nu)(1-2\nu)a}(C_0 + C_1 r^2) -$$

$$- \eta T_2 \int_0^{\frac{\pi}{2}} \sqrt{1 - r^2 \sin^2 \varphi}\, d\varphi + \varepsilon \frac{\alpha E}{1-2\nu}(T_1 + T_2). \tag{18}$$

We will seek a solution of Eq. (15) by using the approximation

$$p(r) = \sum_{k=1}^{N} d_k \varphi_k(r), \tag{19}$$

where $\varphi_k(r)$ are δ-like functions with compact carrier in the neighborhood of the point r_k.

Substituting (19) into (15) and (11), and using the method of collocation, we obtain a system of N + 2 algebraic equations with N + 2 unknowns d_k ($k = \overline{1, N}$), C_0, and C_1. For construction of a program we take single functions as the $\varphi_k(r)$. In Fig. 1 we show graphs of the behavior of the reaction pressure under the base of a circular plate, for an external load q(r) = 1 (see Fig. 1a) and q(r) = 2 − r (see Fig. 1b). All the calculations were carried out for the following values of the parameters appearing in Eq. (15):

$$E = E_0 = 2000\ \text{kG/mm}^2;\ \nu = 0.17;\ \nu_0 = 0.2;\ h = 0.2\text{m};\ a = 1\text{m};$$
$$T_1 = T_2 = 100^\circ;\ \alpha = 1\cdot 10^{-5},\ \alpha_0 = 5\cdot 10^{-6}.$$

The numerical calculations show that taking account of the deformability of the plate over the thickness leads to stable numerical algorithms, and also to the redistribution of the reaction pressure in comparison with the classical results for the bending of a circular plate. Then the parameter ε in Eq. (15) can be considered a regularizing parameter in the solution of an integral equation of the first kind, corresponding to the problem of thermoelastic bending of a circular plate on an elastic base in the classical formulation. The calculations that were given above enable us to give a concrete sense to it.

NOTATION

σ_z, component of the stress tensor; q, external load; p, reaction pressure; T_1, T_2, temperatures of the upper and lower planes of the plate; 2h, height of the plate; $2a$, diameter of the plate; u_r, u_z, components of the displacement vector in cylindrical coordinates; Δ, Laplace operator in cylindrical coordinates; Ψ, thermoelastic potential; α, α_0, coefficients of linear expansion of the plate and the base; λ, μ, Lamé coefficients; E, Young's modulus; ν, Poisson ratio; w_{cr}, crumpling of the plate; w, middle surface of the plate; w_0, sag of the boundary points of the half-space; Φ, thermoelastic potential.

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